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Augmented spaces, two-level methods, and stabilizing subgrids

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SUMMARY

Starting from the already known relationship between stabilized methods, augmented spaces and residual-free bubbles (RFB), the paper introduces a possible way of mimicking the effect of RFB just by constructing a suitable subgrid and then solving the standard Galerkin equations on the modified grid. Concentrating on the model problem of linear convection-dominated equations, we give sufficient conditions on the subgrid that ensure stability, and error bounds of the same type of standard stabilizing procedures. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: stabilized methods; residual-free bubbles; Stabilizing subgrids; convection-dominated equations

1. INTRODUCTION

In recent times, two-level methods are becoming popular in a wide variety of applications. Sometimes they can be used to take advantage of parallel computers, as in domain decomposition methods (see for instance, the series of proceedings of the yearly conference in domain decomposition methods, visiting [1]). Other times, they are used in order to take into account small-scale effects, as for instance when dealing with composite materials having a fine structure (see References [2–4], and the review [5] with the references therein), or when dealing with Helmholtz equations at high frequency [6, 7]. They are also used in *a posteriori* error analysis (see e.g. References [8–10], and the references therein). Finally, they are often also used to stabilize finite element formulations that lack the necessary stability properties, as for convection-dominated flows or Stokes problems [11–13]. In many cases, they are not *seen* as two-level methods, but, as we shall see, they fit rather easily into this category.

The first goal of this paper will indeed be to indicate a general framework that can be seen as a generalization of the augmented space method, in order to include a wide class of these tricks, used for dealing with subscales, into a unified approach.

The second, and main goal of the paper, is to show that within this approach one can set suitable conditions on the subgrids that ensure the optimal performance of the corresponding

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two-level method. We shall do that in the particular case of advection-dominated scalar equations, where much is known (see e.g. References [14–17]), so that the quality of the results can be evaluated in a sharper way. In particular, we shall see that a certain number of stabilized methods can actually be interpreted just as a way of choosing a suitable subgrid, and then applying the usual Galerkin framework (and computer programs). In other words, one can stabilize the problem just by choosing the subgrid. This clearly can also be used in self-adaptive methods.

It would be very interesting to study the possible extensions of this approach to other problems, including more complicated fluid flows, or also problems of different applicative nature.

2. THE MODEL PROBLEM

In order to describe the general idea, we take a simple model problem, or, rather, a class of them. We assume that Ω is a polygon in $\mathbb{R}^2$, and we set

$$V := H_0^1(\Omega)$$

We then consider a bilinear form $(u,v) \to \mathscr{L}(u,v)$ defined as

$$\mathscr{L}(u,v) := \int_\Omega \sum_{i=1}^{2}\left(\sum_{j=1}^{2} a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + ub_i(\mathbf{x})\frac{\partial v}{\partial x_i} + c_i(\mathbf{x})v\frac{\partial u}{\partial x_i}\right) + d(\mathbf{x})uv\,\mathrm{d}\mathbf{x} \tag{2}$$

where clearly $\mathbf{x} = (x_1, x_2)$. The coefficients a_{ij}, b_j, c_i, d are supposed to be smooth functions of $\mathbf{x}$ in Ω. This will easily imply the continuity of the bilinear form $\mathscr{L}$ on $V \times V$, that is

$$\exists M \text{ such that } \mathscr{L}(u,v) \leqslant M\|u\|_V\|v\|_V, \quad \forall u,v \in V \tag{3}$$

To simplify the exposition, we also assume that the bilinear form $\mathscr{L}$ is V-elliptic:

$$\exists \alpha > 0 \text{ such that } \mathscr{L}(v,v) \geqslant \alpha\|v\|_V^2 \quad \forall v \in V \tag{4}$$

For a given right-hand side f, say, in $L^2(\Omega)$, we then consider the variational problem

$$\text{find } u \in V \text{ such that } \mathscr{L}(u,v) = (f,v) \quad \forall v \in V \tag{5}$$

where, as usual, (,) stands for the $L^2(\Omega)$ inner product. It is clear that, thanks to (4), problem (5) has a unique solution. In different applications, (5) can represent a convection-dominated problem, or a problem with a composite material having a fine structure, or just a nice elliptic problem where domain decomposition has to be used in order to take advantage of a parallel computer. The approach that follows, however, can rather easily be extended to systems of equations, including indefinite ones that can be found, for instance, in applications to mixed methods.

In the following sections we shall employ the usual notation for Sobolev norms $\|\cdot\|_{s,\Omega}$ and seminorms $|\cdot|_{s,\Omega}$ (see e.g. Reference [18]).

3. THE GENERAL IDEA

The general idea behind the class of methods we have in mind can be roughly described as follows. We consider a splitting of Ω in a finite number of subpolygons Ω_k $(k=1,\ldots,K)$ in such a way that

$$\bigcup_k \bar{\Omega}_k = \bar{\Omega} \quad \text{and} \quad \Omega_r \cap \Omega_s = \emptyset \ \text{ for } r \neq s \tag{6}$$

In (6) each Ω_k is supposed to be open, and $\bar{\Omega}_k$ represents its closure. Then we set

$$\Sigma := \bigcup_k \partial\Omega_k \tag{7}$$

and we denote by Φ the space of traces on Σ of the functions of V, that is

$$\Phi := \{g \in L^2(\Sigma) \text{ such that } \exists v \in V, v_{|\Sigma} = g\} \tag{8}$$

Then we consider a *finite dimensional* subspace

$$\Phi_H \subset \Phi \quad \text{with } N := \dim(\Phi_H) \tag{9}$$

and the *infinite-dimensional* subspace V_H of V made by the functions in V whose traces on Σ belong to Φ_H, that is

$$V_H := \{v \in V \text{ such that } v_{|\Sigma} \in \Phi_H\} \tag{10}$$

We can now consider the approximate problem:

$$\text{find } u_H \in V_H \text{ such that } \mathscr{L}(u_H, v_H) = (f, v_H) \quad \forall v_H \in V_H \tag{11}$$

It is clear from (4) that problem (11) also has a unique solution. In many applications, the decomposition (6) will be made of triangles, with the usual *compatibility conditions* (namely, for all r and s (with $r \neq s$) the intersection $\bar{\Omega}_r \cap \bar{\Omega}_s$ must be either a common vertex or a common edge or empty). Then, we might choose a finite element space V_P (the *subjacent Polynomial space*) and define Φ_H as the space spanned by the traces of V_P on Σ. In these cases, the stabilizing effects of passing from V_P to V_H are well known. See for instance References [16, 17] for the case of advection-dominated problems. In other cases, however, the structure can be much more complicated. We might for instance have a grid on Σ, and take Φ_H as the set of functions that are continuous on Σ, vanishing on $\Sigma \cap \partial\Omega$ and piecewise polynomial on the given grid. Note that, in this case, the Ω_k's do not need to be triangles or quadrilaterals, and even if they are we do not need compatibility conditions among them. In these cases, there will be no obvious starting space V_P. In other cases the space Φ_H can contain, besides or instead of piecewise polynomials, other functions having suitable properties (exponentials, trigonometric functions, wavelets, or other problem-fitted shapes). During an iterative procedure, these functions might be changed from time to time, using suitable information obtained from the previous steps. As you can see, the framework is rather general.

In *any* case, it is possible to identify the subspace (of *bubbles*) V_B which can simply be defined as

$$V_B := \prod_k H_0^1(\Omega_k) \subset V \equiv H_0^1(\Omega) \tag{12}$$

We can then identify another subspace V_L made of functions v_L in V_H such that

$$\mathscr{L}(v_L, v_{\rm B}) = 0 \quad \forall v_{\rm B} \in V_{\rm B} \tag{13}$$

If L is the differential operator associated with the bilinear form $\mathscr{L}$, the elements of V_L are local solutions of the partial differential equation

$$Lv_L = 0 \quad \text{in } \Omega_k \tag{14}$$

for all k, and having traces on Σ that belong to Φ_H. It is clear that

$$V_H \equiv V_L \oplus V_{\rm B} \tag{15}$$

In some cases it will also be convenient to identify a third subspace, V_{L^*}, made of functions v_{L^*} in V_H such that

$$\mathscr{L}(v_{\rm B}, v_{L^*}) = 0 \quad \forall v_{\rm B} \in V_{\rm B} \tag{16}$$

If L^* is the formal adjoint of the operator L, the elements of V_{L^*} are local solutions of the partial differential equation

$$L^* v_{L^*} = 0 \quad \text{in } \Omega_k \tag{17}$$

for all k, also with traces in Φ_H. It is clear that together with (15) we also have

$$V_H \equiv V_{L^*} \oplus V_{\rm B} \tag{18}$$

We also point out that both V_L and V_{L^*} are finite dimensional, and $\dim(V_L) \equiv \dim(V_{L^*}) \equiv \dim(\Phi_H) \equiv N$.

Given the right-hand side f we can finally consider the particular solution $u_{\rm B}^f \in V_{\rm B}$ such that

$$\mathscr{L}(u_{\rm B}^f, v_{\rm B}) = (f, v_{\rm B}) \quad \forall v_{\rm B} \in V_{\rm B} \tag{19}$$

In strong form, $u_{\rm B}^f$ will be the solution, in every Ω_k, of the boundary value problem

$$Lu_{\rm B}^f = f \text{ in } \Omega_k \quad u_{\rm B}^f = 0 \text{ on } \partial\Omega_k \tag{20}$$

We have then the following theorem.

Theorem 1

Let u_H be the unique solution of (11), and let $u_H = u_L + u_{\rm B}$ be its decomposition according to (15). Then $u_{\rm B}$ coincides with the unique solution $u_{\rm B}^f$ of (19), and u_L can be characterized as the unique solution of either one of the following problems:

$$\text{find } u_L \in V_L \text{ such that } \mathscr{L}(u_L, v_L) + \mathscr{L}(u_{\rm B}, v_L) = (f, v_L) \quad \forall v_L \in V_L \tag{21}$$

or

$$\text{find } u_L \in V_L \text{ such that } \mathscr{L}(u_L, v_{L^*}) = (f, v_{L^*}) \quad \forall v_{L^*} \in V_{L^*} \tag{22}$$

Proof
It is clear from (4) that both (11) and (21) have a unique solution. Let u_H be the solution of (11) and let $u_H = u_L + u_{\rm B}$ be its (unique) decomposition according to (15). Using definition (13) and then (11) for $v_H = v_{\rm B}$ we have

$$\mathscr{L}(u_{\rm B}, v_{\rm B}) = \mathscr{L}(u_L, v_{\rm B}) + \mathscr{L}(u_{\rm B}, v_{\rm B}) = \mathscr{L}(u_H, v_{\rm B}) = (f, v_{\rm B}) \quad \forall v_{\rm B} \in V_{\rm B} \tag{23}$$

which implies that $u_{\rm B}$ coincides with the unique solution $u_{\rm B}^f$ of (19). Then we can take $v_H = v_L$ in (11) and obtain

$$(f, v_L) = \mathscr{L}(u_H, v_L) = \mathscr{L}(u_L + u_{\rm B}, v_L) \quad \forall v_{\rm B} \in V_L \tag{24}$$

telling us that u_L coincides with the unique solution of (21).

We still have to prove that u_L can also be characterized as the solution of (22), and that such solution is unique. Using $u_H = u_L + u_{\rm B}^f$ and $v_H = v_{L^*}$ in (11), and using (16) we immediately have that u_L solves (22). Let now $\widetilde{u_L}$ be another possible solution, in V_L, of (22). It is easy to see that then $\widetilde{u_H} := \widetilde{u_L} + u_{\rm B}^f$ verifies (11) for all $v_{L^*} \in V_{L^*}$ and for all $v_{\rm B} \in V_{\rm B}$. Using (18) we have then that $\widetilde{u_H}$ verifies (11) for all v_H in V_H. As (11) has a unique solution, we conclude that $\widetilde{u_H} \equiv u_H$ and then $\widetilde{u_L} \equiv u_L$, thanks to (15). Hence the uniqueness of the solution of (22) is also proved. □

In the case where one has a *subjacent polynomial space* V_P, one can present the problem in another, slightly different way. Indeed, assuming for simplicity that $V_P \cap V_{\rm B} = \emptyset$, we can now split $V_H = V_P \oplus V_{\rm B}$, and, accordingly, $u_H = u_P + u_{{\rm B}P}$. Then $u_{{\rm B}P}$ solves

$$\mathscr{L}(u_{{\rm B}P}, v_{\rm B}) = -\mathscr{L}(u_P, v_{\rm B}) + (f, v_{\rm B}) \quad \forall v_{\rm B} \in V_{\rm B} \tag{25}$$

that can be written, shortly, as

$$u_{{\rm B}P} = L_{\rm B}^{-1}(f - Lu_P) \tag{26}$$

This, inserted into

$$\mathscr{L}(u_P, v_P) + \mathscr{L}(u_{{\rm B}P}, v_P) = (f, v_P) \quad \forall v_P \in V_P \tag{27}$$

gives

$$\mathscr{L}(u_P, v_P) - (L_{\rm B}^{-1} u_P, L^* v_P) = (f, v_P) - (L_{\rm B}^{-1} f, L^* v_P) \quad \forall v_P \in V_P \tag{28}$$

which could be considered as *another* equivalent way of writing the same problem (11), or (21), or (22). Notice that, in particular, we have $L_{\rm B}^{-1} f \equiv u_{\rm B}^f$ as defined in (19).

Methods of these types are found at several occurrences in the literature. For instance, for convection-dominated problems one can see References [14, 19], and the references therein for methods in formulation (21) or (22), while formulation (28) can be found in Reference [20] and its equivalence with stabilized methods as *SUPG* (see References [21–24]) is made clear in Reference [25]. Formulations of type (21) or (22) can also be found, at a more abstract level but for one-dimensional problems, in Reference [26] and also, in more recent times, in References [3, 4] for homogenization problems. In some sense, the upscaling technique of References [27, 29] can also be seen in this framework, although it uses the mixed formulation as a starting point and hence does not enter directly the present assumptions. See Reference [30] for a more general setting that includes the upscaling methods. Apart from one-dimensional cases (where they all give back the exact solution, provided one solves exactly the differential equation in each subdomain), all these methods require a suitable approximation for the solutions of the problems inside each subdomain, as we shall see below in more detail. A similar point of view could also be taken when looking at domain decomposition problems, where (22) would represent a sort of *continuous Schur complement* that needs however, one way or another, to be discretized.

Indeed, if we consider the problem of the actual solution of all these equivalent formulations, several observations are in order. First of all, problem (19) is infinite dimensional, and therefore its solution is, in general, out of reach. In some cases, however, one might think that the knowledge of the traces of u_L could provide enough information. However, even if problem (22) is actually finite dimensional, it is not solvable in practice. Indeed, in order to solve it on a computer, we should first choose a basis $\{\psi^{(i)}\}$ $(i=1,\dots,N)$ in Φ_H (this is not so difficult) and then associate to it a basis $\{v_L^{(j)}\}$ $(j=1,\dots,N)$ in V_L and a basis $\{v_{L^*}^{(i)}\}$ $(i=1,\dots,N)$ in V_{L^*}, defined by

$$v_L^{(j)}=\psi^{(j)} \text{ on } \Sigma \quad \text{and} \quad Lv_L^{(j)}=0 \quad \text{in } \Omega_k,\ \ (j=1,\dots,N:k=1,\dots,K) \tag{29}$$

and, respectively,

$$v_{L^*}^{(i)}=\psi^{(i)} \text{ on } \Sigma \quad \text{and} \quad L^*v_{L^*}^{(i)}=0 \text{ in } \Omega_k,\ \ (i=1,\dots,N\!: k=1,\dots,K) \tag{30}$$

Then, we can express u_L, as $u_L=\sum_j U_j v_L^{(j)}$ and reduce (22) to the linear system of equations

$$\sum_{j=1}^{N} U_j \mathscr{L}(v_L^{(j)},v_{L^*}^{(i)})=(f,v_{L^*}^{(i)}) \quad \forall i=1,\dots,N \tag{31}$$

However, in order to compute the coefficients $\mathscr{L}(v_L^{(j)},v_{L^*}^{(i)})$ of the matrix in (31), we need to know the values of the $v_L^{(j)}$ and $v_{L^*}^{(i)}$ inside each Ω_k, that requires the solutions of the boundary value problems (29) and (30); and this cannot be obtained in practice. Clearly, we have to resort to some approximate solution. It would be nice, however, to have guidelines that indicate the necessary degree of accuracy that such approximate solution must have.

The same problem arises with the formulation (28). Indeed, expressing now u_P as $u_P=\sum_j U_j v_P^{(j)}$ we should now compute

$$\sum_{j=1}^{N} U_j \mathscr{L}(v_P^{(j)},v_P^{(i)})=(L_{\rm B}^{-1}v_P^{(j)},L^*v_P^{(i)})=(f,v_P^{(i)})-(u_{\rm B}^f,L^*v_P^{(i)}) \quad \forall i=1,\dots,N \tag{32}$$

which again requires the (approximate) solution of the local problems defining $L_{\rm B}^{-1}v_P^{(i)}$ for each j, and $u_{\rm B}^f$. In these cases, having understood the *stabilizing effect* of the additional term appearing in the stiffness matrix of (32), that is $-(L_{\rm B}^{-1}u_P,L^*v_P)$, the efforts have been concentrated mostly in providing approximate solutions of (25) that reproduced *the same stabilizing effect*; see for instance References [14, 31–33]. In particular, when V_P is made of piecewise linear functions, we have that the stabilized problem corresponds exactly to the SUPG method, with a specific value for the stabilizing parameter τ. An approximate solution will produce the same method with a different value of τ. One could then use the theory of SUPG methods (see e.g. References [15, 34]) to get the proper conditions on τ, and hence, backward, on the quality of the approximation. This, however, apart from working only in particular cases, seems somehow *unfair*.

In the next section we are going to follow a different approach. We suppose that in each element Ω_k we have a subgrid, and a finite element space on this subgrid. The discretized solutions of the local problems are then obtained by the *standard* Galerkin finite element approximation. We want to see if we can prescribe reasonable conditions on these finite element (subgrid) spaces, in order to preserve, in a sense to be made precise, the accuracy that was (ideally) obtainable by solving (22). Unfortunately, we will not be able to do that for a completely general problem, but we will have to consider a simplified advection-dominated case. We hope, however, that this might be a first step towards more general results.

4. THE CHOICE OF THE SUBGRID

As announced at the end of the last section, we are now going to consider a particular case of (2). In this particular case, we shall introduce sufficient conditions on the subgrid in order to preserve the quality of the *a priori* error bounds.

More precisely, we shall make the following assumptions on the bilinear form $\mathscr{L}$:

$$\mathscr{L}(u,v) = \varepsilon\mathscr{L}_{\rm s}(u,v) + \mathscr{L}_{\rm a}(u,v) \tag{33}$$

where $\mathscr{L}_{\rm s}(u,v)$ is a bilinear symmetric form on $V \times V$ satisfying

$$|v|^2_{1,\Omega} \leqslant \mathscr{L}_{\rm s}(v,v) \leqslant M_{\rm s}|v|^2_{1,\Omega} \quad \forall v \in V \tag{34}$$

representing the diffusive term, while $\mathscr{L}_{\rm a}$ is a skew-symmetric bilinear form on $V \times V$ satisfying

$$\mathscr{L}_{\rm a}(u,v) \leqslant M_{\rm a}\|u\|_{0,\Omega}|v|^2_{1,\Omega} \quad \forall u,v \in V \tag{35}$$

representing the convective term. Finally, ε is a small parameter. We obviously assume that some characteristic length of Ω (for instance its diameter) has been scaled to 1. It is not difficult to check that the present case is a particular case of (2), that can he obtained for instance by making very mild assumptions on the coefficients a_{ij}, taking d and all b_i's equal to zero and assuming the convective term $\mathbf{c} = (c_j)$ to have zero divergence in Ω.

Before discussing the choice of the subgrid, we first analyse the *a priori* error estimates for problem (11). Following essentially [16], we set

$$e_H := u - u_H \quad \text{and} \quad \eta_H := u - u_H^i \tag{36}$$

where u_H^i is any approximation of u in V_H. We immediately note that

$$e_H - \eta_H \in V_H \tag{37}$$

so that by Galerkin orthogonality we have

$$\mathscr{L}(e_H, e_H - \eta_H) = 0 \tag{38}$$

Using now (34) and (33), then (38), then again (33) and (34), we have

$$\begin{aligned}\varepsilon|e_H|_1^2 &\leqslant \mathscr{L}(e_H, e_H) = \mathscr{L}(e_H, \eta_H) = \varepsilon\mathscr{L}_{\rm s}(e_H, \eta_H) + \mathscr{L}_{\rm a}(e_H, \eta_H)\\ &\leqslant \varepsilon M_{\rm s}|e_H|_1|\eta_H|_1 + \mathscr{L}_{\rm a}(e_H, \eta_H)\end{aligned} \tag{39}$$

The trick to estimate $\mathscr{L}_{\rm a}(e_H, \eta_H)$ is now to consider a generic function $\eta_{\rm B}$ in $V_{\rm B}$ and recall that $V_{\rm B}$ is a subspace of V_H, so that Galerkin orthogonality and (33) imply

$$0 = \mathscr{L}(e_H, \eta_{\rm B}) \equiv \varepsilon\mathscr{L}_{\rm s}(e_H, \eta_{\rm B}) + \mathscr{L}_{\rm a}(e_H, \eta_{\rm B}) \tag{40}$$

Then we can use (35), (40), and (34) and write

$$\begin{aligned}\mathscr{L}_{\rm a}(e_H, \eta_H) &= \mathscr{L}_{\rm a}(e_H, \eta_H - \eta_{\rm B}) + \mathscr{L}_{\rm a}(e_H, \eta_{\rm B})\\ &\leqslant M_{\rm a}|e_H|_1\|\eta_H - \eta_{\rm B}\|_0 - \varepsilon\mathscr{L}_{\rm s}(e_H, \eta_{\rm B})\\ &\leqslant M\varepsilon^{1/2}|e_H|_1(\varepsilon^{-1/2}\|\eta_H - \eta_{\rm B}\|_0 + \varepsilon^{1/2}|\eta_{\rm B}|_1)\end{aligned} \tag{41}$$

having also, in the last step, collected $\varepsilon^{1/2}|e_H|_1$, and set $M := \max\{M_{\rm a}, M_{\rm s}\}$. Defining now

$$\|\eta_H\|_{\simeq 1/2} := \sup_{\varepsilon > 0} \inf_{\eta_{\rm B} \in V_{\rm B}} \{\varepsilon^{-1/2}\|\eta_H - \eta_{\rm B}\|_0 + \varepsilon^{1/2}|\eta_{\rm B}|_1\} \tag{42}$$

we immediately have from (41) and (42) that

$$\mathscr{L}_{\rm a}(e_H, \eta_H) \leqslant M\varepsilon^{1/2}|e_H|_1\|\eta_H\|_{\simeq 1/2} \tag{43}$$

that inserted in (39) gives the final estimate

$$\varepsilon^{1/2}|e_H|_1 \leqslant C(\varepsilon^{1/2}|\eta_H|_1 + \|\eta_H\|_{\simeq 1/2}) \tag{44}$$

As discussed in Reference [16], and in the references therein, norm (42) behaves, from the point of view of interpolation error, as a 1/2-norm (hence the name we adopted here). See however [35] for a much more detailed analysis of these types of norms. Assuming that H is a typical length associated with the size of the Ω_k's, and assuming that, for some integer $s \geqslant 1$, we have the interpolation errors

$$|\eta_H|_{r,\Omega} \leqslant CH^{s+1-r}\|u\|_{s+1,\Omega} \quad r = 0, 1 \tag{45}$$

we have then the *usual* error estimate (see e.g. References [15, 34])

$$\varepsilon^{1/2}|e_H|_{1,\Omega} \leqslant C(\varepsilon^{1/2}H^s + H^{s+1/2}) \tag{46}$$

We point out that, in general, the norm appearing in (45) will depend on ε, unless, by shear luck, the solution has a sufficient degree of smoothness independent of ε. This implies that the constant C appearing in (46) will also depend on ε, unless the solution is smooth. This however is the typical feature of this kind of estimates (see again, for instance, References [15, 34] and the references therein), in which we play the game that the solution is smooth, and we just want to see the degree of accuracy that one would obtain, in this case, with all the other constants independent of ε.

We also notice that, with the same argument as in (41), we easily have, for every $w \in V$ and for every $\eta_{\mathrm{B}} \in V_{\mathrm{B}}$

$$\mathscr{L}_{\mathrm{a}}(e_H, \eta) = \mathscr{L}_{\mathrm{a}}(e_H, \eta - \eta_{\mathrm{B}}) + \mathscr{L}_{\mathrm{a}}(e_H, \eta_{\mathrm{B}}) \leqslant M\varepsilon^{1/2}|e_H|_1 \|\eta\|_{\simeq 1/2} \quad (47)$$

that together with (46) produces a norm of the advective part of the error in the dual norm of $\|\cdot\|_{\simeq 1/2}$. In practical cases, see always [16], this in turn produces the usual L^2 estimate for the advective part of the error, of the type

$$H^{1/2}\|\mathbf{c} \cdot \nabla e_H\|_{0,\Omega} \leqslant C(\varepsilon^{1/2}H^s + H^{s+1/2}) \quad (48)$$

Our target is now to give sufficient conditions on the subgrid discretization in order to preserve the error estimates (46) and (48). For this, we assume that we are given a finite-dimensional subspace $V_H^h \subset V_H$, and we consider the *fully discretized problem*

$$\text{find } u^h \in V_H^h \text{ such that } \mathscr{L}(u^h, v^h) = (f, v^h) \quad \forall v^h \in V_H^h \quad (49)$$

We would like to have, for problem (49), *a priori* error estimates of type (46)–(48). For this, we have to introduce suitable subspaces of V_H^h, as we did before for V_H.

We set

$$V_{\mathrm{B}}^h := V_H^h \cap V_{\mathrm{B}} \quad (50)$$

$$V_L^h := \{v_L^h \in V_H^h \text{ such that } \mathscr{L}(v_L^h, v_{\mathrm{B}}^h) = 0 \quad \forall v_{\mathrm{B}}^h \in V_{\mathrm{B}}^h\} \quad (51)$$

and

$$V_S^h := \{v_S^h \in V_H^h \text{ such that } \mathscr{L}_{\mathrm{s}}(v_S^h, v_{\mathrm{B}}^h) = 0 \quad \forall v_{\mathrm{B}}^h \in V_{\mathrm{B}}^h\} \quad (52)$$

To simplify the notation it will also be convenient to set

$$\|v\|_s^2 := \varepsilon \mathscr{L}_{\mathrm{s}}(v, v) \simeq \varepsilon |v|_1^2 \quad (53)$$

We are now ready to introduce our assumptions on the space V_H^h. We explicitly point out, form the very beginning, that our assumptions are only *sufficient* for getting suitable error bounds. So far, they have been taylored for cases where the local dimension of Φ_H is small, so that we can think to use spaces V_{B}^h that have a small dimension as well. We do believe that there is room for many future improvements, and the present assumptions should be regarded only as a beginning. Our first assumption will be

Assumption 1

There exists a constant C_1, independent of H, h, and ε such that, for every $w \in V$ the solution $\beta^h \in V_{\mathrm{B}}^h$ of

$$\mathscr{L}(\beta^h, b^h) = \mathscr{L}(w, b^h) \quad \forall b^h \in V_{\mathrm{B}}^h \quad (54)$$

satisfies

$$\|\beta^h\|_s + H^{-1/2}\|\beta^h\|_0 \leqslant C_1(\|w\|_s + H^{1/2}\|w\|_1 + H^{-1/2}\|w\|_0) \quad (55)$$

where, here and in all the sequel, H is some characteristic length associated with the Ω_k's (as it was in (46) and (48)): to simplify the exposition, we can assume once and for all that H is the maximum diameter of the Ω_k's.

Assumption 1 should be regarded in the following way: problem (54) corresponds to solve a discrete problem, in each subdomain, exactly of the same type of the original one. For all these problems we require stability estimates of the type that we expect for the global problem (11) (see for instance the estimates (46) and (48)).

We shall come back in a while to discuss possible sufficient conditions that can ensure (55). We first indicate the use that we are going to make of it.

For that we introduce a suitable interpolant of the exact solution u, that will allow an easier derivation of error estimates. We start first by defining u_i^h as the usual interpolant of u in V_H^h. Then we define a new interpolant, $u_{\rm I}^h$ as follows

$$u_{\rm I}^h = u_i^h \quad \text{on } \Sigma \quad \text{and} \quad \mathscr{L}(u_{\rm I}^h, b^h) = \mathscr{L}(u, b^h) \quad \forall b^h \in V_{\rm B}^h \tag{56}$$

Assumption 1 allows us to compare the distance $\|u - u_{\rm I}^h\|$ with the corresponding $\|u - u_i^h\|$.

Theorem 2

Let Assumption 1 hold, let u be a given function in V, and u_i^h be a given function in V_H^h. Assume finally that $u_{\rm I}^h$ is constructed as in (56). Then there exists a constants $C_{\rm I}$ independent of u, u_i^h, H, h, and ε such that

$$\|u - u_{\rm I}^h\|_s + H^{-1/2}\|u - u_{\rm I}^h\|_0 \leqslant C_{\rm I}(\|u - u_i^h\|_s + H^{1/2}\|u - u_i^h\|_1 + H^{-1/2}\|u - u_i^h\|_0) \tag{57}$$

Proof

From (56) we have that u_I^h must have the form $u_{\rm I}^h = u_i^h + \beta^h$, where $\beta^h \in V_{\rm B}^h$ is determined by

$$\mathscr{L}(u_i^h + \beta^h, b^h) = \mathscr{L}(u, b^h) \quad \forall b^h \in V_{\rm B}^h \tag{58}$$

that is

$$\mathscr{L}(\beta^h, b^h) = \mathscr{L}(u - u_i^h, b^h) \quad \forall b^h \in V_{\rm B}^h \tag{59}$$

The proof follows then immediately from (55) using the triangle inequality. □

Essentially, we are requiring that the new interpolant u_I^h defined in (56) is as good as the traditional interpolant u_i^h.

We come back now to the problem of finding sufficient conditions on the subgrid that can ensure (55). A first possibility, rather crude but quite useful in simple cases (for instance when the subgrid contains only one node per element, or just a few) is the following one:

$$\exists C_1' > 0 \text{ such that } \|b^h\|_0 \leqslant C_1' H^{1/2} \|b^h\|_s \quad \forall b^h \in V_{\rm B}^h \tag{60}$$

In the simplest case where we have a poor subgrid, consisting of just one internal node in each element Ω_k, condition (60) is essentially equivalent to (55). Indeed, considering for simplicity a case in which the coefficients in (2) are constant, and w in (54) is linear, assuming that the shape of the bubble b_k is such that, in each Ω_k

$$\|b_k\|_{0,\Omega_k} \simeq |\Omega_k|^{-1/2} \int_{\Omega_k} b_k \, \mathrm{d}\mathbf{x} \tag{61}$$

we can write, in each $\Omega_k, \beta^h = \mu b_k$ and use (54) to determine μ, obtaining

$$\mu = \frac{L_a w \int_{\Omega_k} b_k \,\mathrm{d}\mathbf{x}}{\|b_k\|_s^2} \simeq \|L_a w\|_{0,\Omega} \frac{\|b_k\|_0}{\|b_k\|_s^2} \tag{62}$$

that gives

$$\|\beta^h\|_0 \simeq \|L_a w\|_{0,\Omega_k} \frac{\|b_k\|_0^2}{\|b_k\|_s^2} \tag{63}$$

so that to get (55) we must have (60).

Inequality (60) should be compared with the usual Poincaré inequality, that would give

$$\|b^h\|_0 \leqslant CH|b^h|_1 \quad \forall b^h \in V_B^h \tag{64}$$

In d dimensions, for a 'normally shaped' bubble b^h with maximum value equal to 1, we expect $\|b^h\|_0$ to behave like $H^{d/2}$ and $|b^h|_1$ to behave like $H^{d/2-1}$. Here we are dealing with a two-dimensional problem; roughly speaking, in order to fulfil (60) we must have that, in each macroelement $\Omega_k, |b^h|_1$ behaves as $\varepsilon^{-1/2}H^{1/2}$, instead of being $\simeq 1$. Inequality (60) (that actually would be the same in any dimension) requires therefore that the subgrid nodes are at a distance $\simeq \varepsilon$ (or smaller) from the boundary of the corresponding Ω_k, as it is for instance the case for the pseudo-residual-free bubbles of [31], or for Shishkin meshes [36]. We shall see in a while that, if we have in mind subspaces V_B^h having more than a few degrees of freedom, (60) is too restrictive. However its use is quite easy, and we prefer to start with it rather than with more complicated variants. It is easy to see that (60) indeed implies (55), when $\mathscr{L}$ has the structure described in (33) with (34) and (35). Actually, taking $b^h = \beta^h$ in (54), using (33), (34), and (35), and finally using (60), we obtain

$$\begin{aligned}
\|\beta^h\|_s^2 &= \mathscr{L}(\beta^h, \beta^h) = \mathscr{L}(w, \beta^h) \\
&= \varepsilon\mathscr{L}_s(w, \beta^h) + \mathscr{L}_a(w, \beta^h) \\
&\leqslant M_s\|w\|_s\|\beta^h\|_s + M_a\|w\|_1\|\beta^h\|_0 \\
&\leqslant \|\beta^h\|_s(M_s\|w\|_s + M_a C_1' H^{1/2}\|w\|_1)
\end{aligned} \tag{65}$$

which easily gives the required estimate for $\|\beta^h\|_s$. To estimate $\|\beta^h\|_0$ use again (60).

Another reasonably simple possibility would be to require that

$$\exists \kappa_1 > 0 \text{ such that } \|\beta^h\|_0 \leqslant \kappa_1 H \sup_{b^h \in V_B^h} \frac{\mathscr{L}(\beta^h, b^h)}{\|b^h\|_0} \quad \forall \beta^h \in V_B^h \tag{66}$$

together with

$$\exists \kappa_2 > 0 \text{ such that } \varepsilon\|b^h\|_1 \leqslant \kappa_2\|b^h\|_0 \quad \forall b^h \in V_B^h \tag{67}$$

It is easy to see that (60) and (66) coincide when V_B^h has only one degree of freedom per element. Indeed, in this case

$$H \sup_{b^h \in V_B^h} \frac{\mathscr{L}(\beta^h, b^h)}{\|b^h\|_0\|\beta^h\|_0} = H\frac{\mathscr{L}(\beta^h, \beta^h)}{\|\beta^h\|_0^2} = H\frac{\|\beta^h\|_s^2}{\|\beta_h\|_0^2} \tag{68}$$

On the other hand, also in the more general case, (66) and (67) always ensure (55). Indeed, in the last step of (65), instead of $\|\beta^h\|_0 \leqslant C_1' H^{1/2} \|\beta^h\|_s$, we could use (59) in (66) to obtain the following estimate:

$$\|\beta^h\|_0 \leqslant \kappa_1 H \sup_{b^h \in V_{\mathrm{B}}^h} \frac{\mathscr{L}(w, b^h)}{\|b^h\|_0} \tag{69}$$

and then use (33)–(35), (53), and (67) to obtain, for every $b^h \in V_{\mathrm{B}}^h$

$$\begin{aligned}\mathscr{L}(w, b^h) = \mathscr{L}_{\mathrm{s}}(w, b^h) + \mathscr{L}_{\mathrm{a}}(w, b^h) &\leqslant \varepsilon M_{\mathrm{s}} \|w\|_1 \|b^h\|_1 + M_{\mathrm{a}} \|w\|_1 \|b^h\|_0 \\ &\leqslant \max\{M_{\mathrm{s}}\kappa_2, M_{\mathrm{a}}\} \|w\|_1 \|b^h\|_0\end{aligned} \tag{70}$$

Inserting it into (69) we have

$$\|\beta^h\|_0 \leqslant \kappa_1 H \max\{M_{\mathrm{s}}\kappa_2, M_{\mathrm{a}}\} \|w\|_1 \tag{71}$$

Then, using (71) in the last step of (65) gives

$$\|\beta^h\|_s^2 \leqslant M_{\mathrm{s}} \|\beta^h\|_s \|w\|_s + M_{\mathrm{a}} \kappa_1 H \max\{M_{\mathrm{s}}\kappa_2, M_{\mathrm{a}}\} \|w\|_1^2 \tag{72}$$

that, together with (71), provides the desired bound (55).

We also point out that, unfortunately, (60) will not be satisfied if the subgrid has one or more internal nodes having distance of order H from all the other nodes. In this situation we would indeed be able to construct a function b^h in V_{B}^h with $\|b^h\|_0 \simeq H$ and $\|b^h\|_s \simeq \varepsilon^{1/2}$, making (60) impossible to satisfy with C_1' independent of ε.

Our second assumption will be needed in order to prove error bounds for $\|u - u^h\|$. In order to present it, we shall need however one further piece of notation. To every $v^h \in V_H^h$ we associate in a unique way two other elements of V_H^h, that we call $v_L^h(v^h)$ and $v_S^h(v^h)$ (or, shortly, just v_L^h and v_S^h, respectively) by the conditions

$$v_L^h(v^h) = v_S^h(v^h) = v^h \text{ on } \Sigma \quad \text{and} \quad v_L^h(v^h) \in V_L^h, v_S^h(v^h) \in V_S^h \tag{73}$$

where V_L^h and V_S^h are defined in (51) and (52), respectively.

Assumption 2

$$\exists C_2 > 0 \text{ such that } \forall v^h \in V_H^h \text{ we have } \|L_{\mathrm{a}} v_S^h(v^h)\|_0 \leqslant C_2 H^{-1/2} \|v_L^h(v^h)\|_{\mathrm{s}} \tag{74}$$

where clearly L_{a} is the (advective) operator associated with the bilinear form $\mathscr{L}_{\mathrm{a}}$ in (33).

At first sight, Assumption 2 might seem rather obscure. A possible way of looking at it is the following: we are comparing the local discrete solutions of two different problems, with the same boundary data. Indeed, v_S^h and v_L^h have the same value on the boundary of each Ω_k, and represent the discrete solutions, on the given subgrid, of $L_{\mathrm{s}} v = 0$ and $Lv = 0$, respectively, where clearly L_{s}, in agreement with (33), denotes the symmetric part of the operator L. In both sides of (74) we have terms including first derivatives, but on the right-hand side we have a term that behaves like $H^{-1/2}\varepsilon^{1/2}$, that is much smaller than 1 in the interesting cases.

Assumption 2 requires that the subgrid is such that the discrete solution of the *bad problem* ($Lv = 0$) comes out to be *bad enough so* that its $\|\cdot\|_1$ norm is big enough to compensate for the smallness of $H^{-1/2}\varepsilon^{1/2}$. However, a sufficient condition for (74) to hold is to have

$$\|L_{\mathrm{a}} v_S^h\| \leqslant C_3 H^{-1/2} \sup_{b^h \in V_{\mathrm{B}}^h} \frac{\mathscr{L}_{\mathrm{a}}(v_L^h, b^h)}{\|b^h\|_s} \quad \forall v^h \in V_H^h \tag{75}$$

for some positive constant C_3, where v_S^h and v_L^h are defined, starting from v^h, as in (73). Indeed, owing to the properties of functions v_L^h we have, for all $b^h \in V_{\mathrm{B}}^h$,

$$\mathscr{L}_{\mathrm{a}}(v_L^h, b^h) = -\varepsilon \mathscr{L}_{\mathrm{s}}(v_L^h, b^h) \leqslant M_{\mathrm{s}} \|v_L^h\|_s \|b^h\|_s \tag{76}$$

Hence (75) implies (74) with $C_2 = C_3 M_{\mathrm{s}}$. We note that, surprisingly enough, a small value of ε is actually *helping* in proving (75) for a given choice of subgrid spaces. Indeed, a small ε will, in general, make the norm $\|b^h\|_s$ smaller (see (53)) in the denominator of (75), without changing $\|L_{\mathrm{a}} v_S^h\|_0$ (that does not depend on ε). In practical cases, the numerator of (75), having fixed b^h and v^h (that is, the values of v_L^h on Σ), also increases when ε becomes smaller. Indeed, we remind that, for a fixed v^h, the value of $v_L^h(v^h)$, as defined in (73), grows when ε becomes smaller. It seems therefore that, in this approach, the care to be taken for a small ε is all in Assumption 1. On the other hand, for instance in the case of one bubble per element, it might happen that the shape of the bubbles b_k is such that $\|b_k\|_s$, instead of behaving like $H^{1/2}$ (or as $H^{d/2-1/2}$ in d dimensions) as required by (60), is actually bigger. This would correspond, for instance, to having a node whose distance from $\partial\Omega_k$ is *smaller* than ε. Then (74) might be violated, as the denominator in (75) becomes too big. The use of (60) and (74) together seems then to require that the internal node is *exactly* at a distance of order ε from the boundary. This agrees perfectly with the results obtained in Reference [11] in a more particular case.

Remark
One might wonder why we took the pain to introduce v_S^h, and use it in the left-hand side of (74). The reason is simple. If we took v_L^h instead of v_S^h in the left-hand side of (74) we would have obtained a very powerful assumption that is never satisfied, even in the simplest examples (one dimension, constant coefficients, etc.). □

We are now ready to obtain error estimates for problem (49).

Theorem 3
In the same assumptions of Theorem 2, let u and u^h be the solutions of (5) and (49), respectively, and let u_i^h be given in V_H^h. Let moreover u_{I}^h be defined as in (56). Then there exists a constant γ_{s}, independent of u, u^h, u_i^h, H, h, and ε such that

$$\|u - u^h\|_s \leqslant \gamma_{\mathrm{s}}(\|u - u_{\mathrm{I}}^h\|_s + H^{-1/2}\|u - u_{\mathrm{I}}^h\|_0) \tag{77}$$

Proof
We set $e^h := u^h - u_{\mathrm{I}}^h$ and $\eta^h := u - u_{\mathrm{I}}^h$. We notice that $e^h - \eta^h = u^h - u$, so that, by Galerkin orthogonality,

$$\mathscr{L}(e^h - \eta^h, v^h) = 0 \quad \forall v^h \in V_H^h \tag{78}$$

Moreover, for all $b^h \in V_{\rm B}^h$ we have, using (49), (56) and (5)

$$\mathscr{L}(e^h, b^h) = \mathscr{L}(u^h, b^h) - \mathscr{L}(u_{\rm I}^h, b^h) = (f, b^h) - \mathscr{L}(u, b^h) = 0 \tag{79}$$

implying

$$e^h \in V_L^h \text{ (and hence } e_L^h \equiv e^h) \tag{80}$$

that will be used later on. We can now use (53), (78), and (33) to obtain

$$\|e^h\|_s^2 = \mathscr{L}(e^h, e^h) = \mathscr{L}(\eta^h, e^h) = \varepsilon\mathscr{L}_{\rm s}(\eta^h, e^h) + \mathscr{L}_{\rm a}(\eta^h, e^h) \equiv {\rm I} + {\rm II} \tag{81}$$

The bound for I is immediate

$${\rm I} = \varepsilon\mathscr{L}_{\rm s}(\eta^h, e^h) \leqslant M_{\rm s}\|\eta^h\|_s\|e^h\|_s \tag{82}$$

To bound II requires some additional work: first we introduce e_S^h as in (73). We notice immediately that e_S^h turns out to be the *projection* of e^h onto V_S^h in the $\|\cdot\|_{\rm s}$-norm. Indeed for all $v_S^h \in V_S^h$ we have

$$\mathscr{L}_{\rm s}(e^h - e_S^h, v_S^h) = 0 \tag{83}$$

since $e^h - e_S^h$ belongs to $V_{\rm B}^h$ and $\mathscr{L}_{\rm s}$ is symmetric. We deduce that, in particular,

$$\|e_S^h\|_s^2 + \|e^h - e_S^h\|_s^2 = \|e^h\|_s^2 \tag{84}$$

To estimate II we add and subtract e_S^h

$${\rm II} = \mathscr{L}_{\rm a}(\eta^h, e^h) = \mathscr{L}_{\rm a}(\eta^h, e_S^h) + \mathscr{L}_{\rm a}(\eta^h, e^h - e_S^h) \equiv {\rm III} + {\rm IV} \tag{85}$$

and we bound the two pieces separately. Using Cauchy–Schwarz, (74), and finally (80) we obtain

$${\rm III} = \mathscr{L}_{\rm a}(\eta^h, e_S^h) \leqslant \|\eta^h\|_0\|L_{\rm a}e_S^h\|_0 \leqslant \|\eta^h\|_0 C_2 H^{-1/2}\|e_L^h\|_s = \|\eta^h\|_0 C_2 H^{-1/2}\|e^h\|_s \tag{86}$$

In order to bound IV we first notice that, thanks to (73) $e^h - e_S^h$ belongs to $V_{\rm B}^h$. Using (56) we have then

$$\mathscr{L}_{\rm a}(\eta^h, e^h - e_S^h) + \mathscr{L}_{\rm s}(\eta^h, e^h - e_S^h) = \mathscr{L}(\eta^h, e^h - e_S^h) = \mathscr{L}(u - u_{\rm I}^h, e^h - e_S^h) = 0 \tag{87}$$

Now using (87), (34), and (84) we have

$${\rm IV} = \mathscr{L}_{\rm a}(\eta^h, e^h - e_S^h) \leqslant M_{\rm s}\|\eta^h\|_s\|e^h - e_S^h\|_s \leqslant M_{\rm s}\|\eta^h\|_s\|e^h\|_s \tag{88}$$

Collecting (81), (82), (85), (86), and (88) we have

$$\|e^h\|_s^2 \leqslant \|e^h\|_s(2M_{\rm s}\|\eta^h\|_s + C_2 H^{-1/2}\|\eta^h\|_0) \tag{89}$$

and we conclude the proof using the triangle inequality. □

From (74), (80), and (89) we immediately have an estimate on the convective part of the error

$$H^{1/2}\|L_{\rm a}e_S^h\| \leqslant C_2\|e_L^h\|_s = C_2\|e^h\|_s \leqslant \max\{2C_2M_{\rm s}, C_2^2\}(\|\eta^h\|_s + H^{-1/2}\|\eta^h\|_0) \tag{90}$$

Comparing (77) and (90) with the previous results for the corresponding errors for $u - u_H$ (see e.g. (46) and (48)), we see that our assumptions insure errors of the same size.

5. CONCLUSIONS

We have seen a rather general setting that includes many variants of two-level methods that have been developed, more or less independently from each other, for various applications. Many stabilized methods can also be included in this setting. We have seen as well that, for certain problems like convection-dominated flows, the required stabilizing effect can be obtained just with a suitable choice of the subgrid. In particular, we proposed sufficient conditions on the subgrid discretization in order to obtain error estimates of the same quality as one could obtain by solving (ideally) the fine-level equations in an exact way.

The use of conditions of this type in self-adaptive procedures is surely worth investigating, as well as their extension to non-conforming approximations for the subgrid problems, or to other applications.

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